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Stochastic quantum Langevin equation

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Abstract. The stochastic representation of quantum mechanics is used to give a purely stochastic treatment of the quantum Langevin equation. A general coupling involving positions and velocities of the particle and harmonic modes, with arbitrary dependence on the particle's position, is dealt with. Summing over the modes in the path integral and using stochastic integrals, one obtains explicitly the reduced measure, that is, the stochastic process representing the time evolution of the small system in the presence of noise. Contact is made with the operatorial representation, and a relation is established between the correlations of the noise in both representations. A general condition on the coupling constants is given for recovering time locality in the continuum limit, and the effects of renormalisation are re-analysed within this purely stochastic approach.

1. Introduction

Since its introduction in the phenomenological description of Brownian motion (Langevin 1908), Langevin's equation has become a generic description of motion in the presence of noise (Van Kampen 1981). In classical mechanics, this received a precise formulation within the framework of stochastic processes, where the classical (differentiable) trajectory is replaced by a statistical measure on a set of fluctuating (non-differentiable) trajectories (Wiener 1923). The Langevin equation is then better seen as a stochastic differential equation, for which a stochastic differential and integral calculus has been developed (Itô 1951, Schuss 1980), generalising the classical resolution of the equations of motion. Quantum mechanics has made it necessary to revise the description and to formulate a corresponding quantum Langevin equation (Mori 1965, Ford *et al* 1965). This also led to the quantum generalisation of stochastic processes, non-commuting stochastic processes (Lewis and Thomas 1975, Davies 1976), and to the associated calculus (Hasegawa and Streater 1983, Hudson and Parthasarathy 1984). This last development accounts both for fluctuations having their origin in the 'purely' statistical aspects of the problem (the presence of noise), and for fluctuations resulting from 'purely' quantum aspects. Accordingly, the formalism mixes operators and measures in a novel and rather complex way.

Meanwhile, other representations of quantum mechanics were also developed, which progressively established a close connection between quantum mechanics and stochastic processes. The first efficient representation was given by Feynman (1948), in terms of path integrals. Although defined in an intuitive way, the path integrals already represent quantum mechanics by generalising the classical trajectories to

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randomly fluctuating paths. Since then, a sound mathematical basis has been given to path integrals by means of stochastic processes (Glimm and Jaffe 1981). Further developments improved the equivalence between the operatorial and the stochastic representations of quantum mechanics (Nelson 1966) and tended to show that the latter is more than an alternative formal description, but can even constitute a genuine physical approach to quantum phenomena. Indeed, this approach aims at showing that quantum motion is like Brownian motion, where the noise is induced by the vacuum (like in the background field hypothesis (Nelson 1985)).

Path integrals, when used to describe a small system coupled to a bath (Feynman and Vernon 1963) already appear particularly well suited for treating simultaneously pure quantum effects and statistical effects, like tunnelling and dissipation (Caldeira and Leggett 1981). However, one would like to exploit the stochastic background of this formalism and relate it to the statistical nature of the Langevin equation. In view of the equivalence between the stochastic and the operatorial representations of quantum mechanics, the question naturally arises whether the quantum Langevin equation can be formulated within a purely stochastic framework. This would result in a formalism where fluctuations due to the noise induced by the bath and quantum fluctuations are treated in the same way, that is, where only measures appear, instead of measures and operators.

This paper is devoted to this approach. A purely stochastic treatment of the quantum Langevin equation relies on representing both the small system and the bath (taken to be an infinite set of harmonic oscillators) by a stochastic process which will generate all the desired quantum correlation functions, that is, the rigorous form of the related Feynman path integrals. A general coupling between the positions and velocities of both the particle and the modes will be envisaged, so that additional Itô terms will appear (DeWitt 1957, MacLaughlin and Schulman 1971). The formalism of stochastic processes will allow one to deal with them and will also provide the stochastic integrals, which are necessary for expressing the contribution of non-differentiable trajectories. The present treatment will turn out to cover previously studied cases (Caldeira and Leggett 1981, Ford *et al* 1985, Ford and Kac 1986, Nakazawa 1986) and others. A general condition on the coupling constants will also be obtained for recovering a process which evolves locally in time in the limit of a bath consisting of a continuum of modes, with the related feature of two renormalisations affecting the resulting stochastic process.

2. Quantum Langevin path integrals

This section reconsiders a program which has been described by Ford *et al* (1965) (and which, as they note, goes back to Gibbs): consider a small system coupled to a bath, which for the sake of solvability, is supposed to consist of a set (even infinite) of harmonic oscillators; eliminate the degrees of freedom of the bath, up to their initial (final) values; for various statistical distributions of these boundary values, study the resulting statistical properties of the degrees of freedom of the small system. The aim of this approach is to show that a Langevin-type equation (as is proposed on phenomenological grounds) does indeed govern the time evolution of the small system. Ford *et al* solved this problem within the frameworks of classical and quantum mechanics, that is, by solving ordinary and then operatorial equations of motion. Here, the quantum problem will be reconsidered, using a stochastic representation of quantum mechanics. Instead of solving the equations of motion for the bath degrees of freedom, we shall integrate over the latter, in the measure which generates the different correlation functions or Green functions. Before attacking this program, we shall need the correct measure that will describe, in this formalism, the quantum evolution of the whole system.

2.1. *Diffusion processes and stochastic representation*

Let us first consider, quite generally, $N+1$ degrees of freedom, which will be represented by $N+1$ time-dependent random variables $(x^i(t))_{i=0,N}$ obeying a diffusion process, i.e. such that

$$\begin{aligned} &\mathrm{d}x^i = x^i(t+\mathrm{d}t) - x^i(t) \qquad \text{with } \mathrm{d}t \to 0 \\ &\langle \mathrm{d}x^i \rangle_x = b^i(x)\,\mathrm{d}t + o(\mathrm{d}t) \\ &\langle \mathrm{d}x^i\,\mathrm{d}x^j \rangle_x = 2\nu^{ij}(x)\,\mathrm{d}t + o(\mathrm{d}t) \\ &\langle \mathrm{d}x^{i_1}\,\mathrm{d}x^{i_2}\ldots\mathrm{d}x^{i_n} \rangle_x = o(\mathrm{d}t) \end{aligned}$$

where t is the time parameter and $\langle\ \rangle_x$ denotes the expectation value, conditional in $x^i(t)$; $b^i(x)$ is the drift field and $\nu^{ij}(x)$ is the diffusion field. Such a process (with further conditions on b and ν that will be specified later) gives a stochastic representation of the quantum system described by the $N+1$ degrees of freedom x^i (x^0 for the small system and $(x^i)_{i=1,N}$ for the bath). We shall first write the expression of the joint probability $\mathcal{P}$ of the variables $x^i(t)$, in the case when the process is supposed to be Markovian (for the whole set of variables $(x^i)_{i=0,N}$). If $\rho(t_0)$ denotes the probability density of the variables $x^i(t_0)$ (t_0 will go to $-\infty$), the joint probability is then completely determined and given by

$$\begin{aligned} &\mathcal{P} = \exp\left(-I\int_{t_0}^{t_0'} \mathrm{d}S\right)\rho(t_0)\mathcal{D}x \\ &\mathcal{D}x = \prod_t \frac{1}{\sqrt{|2\nu|}} \prod_i \frac{\mathrm{d}x'^i}{\sqrt{\mathrm{d}t}} \\ &x'^i = x^i(t+\mathrm{d}t) \qquad |2\nu| = \det(2\nu^{ij}) \\ &\mathrm{d}S = \tfrac{1}{4}\nu_{ij}\frac{\mathrm{d}x^i\,\mathrm{d}x^j}{\mathrm{d}t} + \gamma_{ijk}(\nu)\frac{\mathrm{d}x^i\,\mathrm{d}x^j\,\mathrm{d}x^k}{\mathrm{d}t} + \beta_i\,\mathrm{d}x^i + \eta_{ijkl}(\nu)\frac{\mathrm{d}x^i\,\mathrm{d}x^j\,\mathrm{d}x^k\,\mathrm{d}x^l}{\mathrm{d}t} \\ &\qquad + \varepsilon_{ij}(\beta,\nu)\,\mathrm{d}x^i\,\mathrm{d}x^j + \delta\,\mathrm{d}t \end{aligned} \tag{1}$$

where

$$\begin{aligned} &\nu_{ij}\nu^{jk} = \delta_{ik} \qquad (\delta \text{ is the Krönecker symbol}) \\ &\gamma_{ijk}(\nu) = \tfrac{1}{24}(\partial_i\nu_{jk} + \partial_j\nu_{ik} + \partial_k\nu_{ij}) \\ &\eta_{ijkl}(\nu) = \tfrac{1}{144}[\partial_i\partial_j\nu_{kl} + \partial_i\partial_k\nu_{jl} + \partial_i\partial_l\nu_{jk} + \partial_j\partial_k\nu_{il} + \partial_j\partial_l\nu_{ik} + \partial_k\partial_l\nu_{ij} \\ &\qquad - \nu_{mn}(\Gamma^m_{ij}\Gamma^n_{kl} + \Gamma^m_{ik}\Gamma^n_{jl} + \Gamma^m_{il}\Gamma^n_{jk})] \\ &\Gamma^i_{jk} = \frac{\nu^{il}}{2}(\partial_j\nu_{lk} + \partial_k\nu_{lj} - \partial_l\nu_{jk}) \qquad \mu^i = -\nu^{jk}\Gamma^i_{jk} \\ &\beta_i = -\tfrac{1}{2}\nu_{ij}(b^j - \mu^j) + \tfrac{1}{2}\partial_i \ln|\nu| \end{aligned}$$

$$\varepsilon_{ij}(\beta,\nu)=\tfrac{1}{8}\partial_t\nu_{ij}+\tfrac{1}{4}(\partial_i\beta_j+\partial_j\beta_i)$$

$$\delta=\tfrac{1}{4}\nu_{ij}(b^i-\mu^i)(b^j-\mu^j)+\tfrac{1}{4}\partial_t\ln|\nu|+\tfrac{1}{2}\sqrt{|\nu|}\partial_i\left(\frac{1}{\sqrt{|\nu|}}(b^i-\mu^i)\right).$$

This follows from the Markov property and the fact that the conditional probability

$$P(x'^i,t'|x^i,t)=\mathrm{e}^{-\mathrm{d}S}\frac{1}{\sqrt{|2\nu|\mathrm{d}t^N}}\qquad(t'=t+\mathrm{d}t)$$

is a solution of the Fokker-Planck equation (Jaekel and Pignon 1985):

$$\partial_{t'}P+\partial'_i[(b'^i-\partial'_j\nu'^{ij})P]=0\qquad \partial'_i=\frac{\partial}{\partial x'^i};\ b'^i=b^i(x'),\ \nu'^{ij}=\nu^{ij}(x').$$

As usual with Markovian processes, these stochastic integrals must be understood as Itô integrals; that is, by definition, for any diffusion variables f and g

$$I\int_{t_0}^{t'_0}f\,\mathrm{d}g=\lim_{n\to\infty}\sum_{i=0}^{n-1}f(t_i)[g(t_{i+1})-g(t_i)]\qquad t_i=t_0+\frac{i}{n}(t'_0-t_0)$$

thus corresponding to non-anticipating functions.

For q an arbitrary function of x and t, the measure can also be rewritten as

$$\mathscr{P}=\psi^*(t'_0)\mathscr{I}\psi(t_0)\qquad \mathscr{I}=\exp\left(-I\int_{t_0}^{t'_0}\mathrm{d}\mathscr{S}\right)\mathscr{D}x$$

$$\psi=\mathrm{e}^q\rho\qquad \psi^*=\mathrm{e}^{-q}.$$

$\mathrm{d}\mathscr{S}=\mathrm{d}S-\mathrm{d}q$ has still the form of $\mathrm{d}S$ in (1), where ν^{ij} is left unchanged (and so are ν_{ij}, γ_{ijk} and η_{ijkl}), β_i has been replaced by $\beta_i-\partial_iq$, (and thus ε_{ij} by $\varepsilon_{ij}-\frac{1}{2}\partial_i\partial_jq$), and δ has been replaced by $\delta-\partial_tq$. Now, if the classical Lagrangian that determines the time evolution of the degrees of freedom is a general quadratic expression in the velocities, i.e. of the following form:

$$\mathscr{L}=\tfrac{1}{2}m_{ij}\dot{x}^i\dot{x}^j-a_i\dot{x}^i-V\qquad(m^{ij}m_{jk}=\delta_{ik})\tag{2}$$

a pure state will be defined as a set of two functions ρ and q (or ψ and ψ^*) satisfying

$$\hbar\partial_t(\bar{q}+\tfrac{1}{4}\partial_t\ln|m|)+\hbar\sqrt{|m|}\partial_i\left(\frac{m^{ij}}{\sqrt{|m|}}(\hbar\partial_j\bar{q}-a_j)\right)$$
$$-\frac{m^{ij}}{2}(\hbar\partial_i\bar{q}-a_i)(\hbar\partial_j\bar{q}-a_j)+V=0\qquad \bar{q}=q-\tfrac{1}{2}\ln|\nu|\tag{3}$$
$$\partial_t\rho-\partial_i\{m^{ij}[\hbar\partial_j(\bar{q}+\tfrac{1}{2}\ln\rho\sqrt{|m|})-a_j]\rho\}=0.$$

With a pure state (ρ,q) will be associated the Markovian diffusion process which has probability density ρ, diffusion field ν^{ij} and drift field b^i given by $2\nu^{ij}=\hbar m^{ij}$, $b^i=\mu^i-m^{ij}(\hbar\partial_j\bar{q}-a_j)$. The second equation of (3) is then the Fokker-Planck equation governing the time evolution of the probability density according to the diffusion process just defined, while the first equation of (3) provides a dynamical evolution according to the classical Lagrangian (2). In this way, for this classical Lagrangian, any related pure state (ψ,ψ^*) or any mixed state (defined by a linear combination

(Jaekel and Pignon 1984)) is associated with a diffusion process having the following joint probability:

$$\mathcal{P} = \hat{\rho}(t_0', t_0)\mathcal{J} \qquad \hat{\rho}(t_0', t_0) = \sum_n \lambda_n \psi_n^*(t_0')\psi_n(t_0)$$

$$\mathcal{J} = \exp\left(-I\int_{t_0}^{t_0'} \mathrm{d}\mathcal{S}\right)\mathcal{D}x \tag{4}$$

$$\hbar\,\mathrm{d}\mathcal{S} = \tfrac{1}{2}m_{ij}\frac{\mathrm{d}x^i\,\mathrm{d}x^j}{\mathrm{d}t} + 2\gamma_{ijk}(m)\frac{\mathrm{d}x^i\,\mathrm{d}x^j\,\mathrm{d}x^k}{\mathrm{d}t} + 2\eta_{ijkl}(m)\frac{\mathrm{d}x^i\,\mathrm{d}x^j\,\mathrm{d}x^k\,\mathrm{d}x^l}{\mathrm{d}t} - a_i\,\mathrm{d}x^i$$
$$+\,\varepsilon_{ij}(-a, m)\,\mathrm{d}x^i\,\mathrm{d}x^j + V\,\mathrm{d}t$$

where $\hat{\rho}$ characterises a mixture, with probabilities λ_n, of states ψ_n (mixed states for both the small system and the modes must be envisaged, if only to include thermal states). In particular, the correlation functions of this state will be given by

$$\left\langle \prod_i x(t_i)\right\rangle = \int \prod_i x(t_i)\mathcal{P}.$$

The path integral representation of quantum mechanics (DeWitt 1957) is then obtained by making Wick's rotation:

$$t \to -\mathrm{i}t \qquad a_i \to \mathrm{i}a_i. \tag{5}$$

In this limit, ψ (ψ^*) becomes the wavefunction of the state (its complex conjugate), which satisfies the Schrödinger equation (3) corresponding to the (classical) Lagrangian (2). The correspondence between a classical Lagrangian $\mathcal{L}$ and a diffusion process $\mathcal{P}$ is achieved by requiring that $\hbar\,\mathrm{d}\mathcal{S}$ should have $\mathcal{L}\,\mathrm{d}t$ as a limit when the random variables become differentiable, i.e. when $\mathrm{d}x^i \sim \mathrm{d}t$ and $\mathrm{d}x^i\,\mathrm{d}x^j \ldots \mathrm{d}x^k = \mathrm{o}(\mathrm{d}t)$ (classical limit). This, after Wick's rotation (5), provides the path integral representation of the quantum system governed by the Lagrangian defined in (2). The correlation functions of the process then identify with the time-ordered quantum correlation functions of the degrees of freedom $x^i(t)$:

$$\left\langle \prod_i x(t_i)\right\rangle = \left\langle \psi \left| T\prod_i \hat{x}(t_i)\right| \psi\right\rangle.$$

In this limit, the correspondence between quantum states and stochastic processes also identifies with Nelson's (Nelson 1966, Guerra 1981). The strategy in the following will be first to apply the program defined at the beginning of this section to the general measure (4) associated with a well chosen classical Lagrangian, and then to perform Wick's rotation at the end.

2.2. Reduced measure

The classical Lagrangian will be chosen to describe the time evolution of a small system, with degree of freedom denoted by x^0, coupled to a set of harmonic oscillators, denoted by x^k, $k = 1, N$ (with N going to infinity), and playing the role of the bath. The coupling will be assumed to be at most quadratic in the velocities, $\dot{x}^0$ and $\dot{x}^k$ (so as to remain within diffusion processes), and linear with respect to the harmonic modes' positions and velocities, x^k and $\dot{x}^k$. But the coefficients will be allowed to depend arbitrarily on the small system's variable, x^0. This general quadratic coupling will include previously studied cases (Ford and Kac 1986, Nakazawa 1986, Caldeira and Leggett 1981) with that of a particle coupled to the electromagnetic field (Ford *et al* 1985). With these assumptions, the most general classical Lagrangian will be written as

$$\mathscr{L}=\tfrac{1}{2}m_0(\dot{x}^0)^2-V+\sum_{k=1}^{N}\left[-(\dot{\alpha}_k+\gamma_k)\dot{x}^k-(\dot{\beta}_k+\delta_k)x^k+\tfrac{1}{2}(\dot{x}^k)^2-\tfrac{1}{2}\omega_k^2(x^k)^2\right] \tag{6}$$

where m_0, V, α_k, β_k, γ_k, δ_k are functions of t and x^0 only, and

$$\dot{f}=\frac{\mathrm{d}f}{\mathrm{d}t}=\partial_t f+\sum_{i=0}^{N}\partial_i f\dot{x}^i.$$

The joint probability $\mathscr{P}$ of the process will be written according to (4) as

$$\mathscr{P}=\hat{\rho}(t_0',t_0)\mathscr{I}\qquad \mathscr{I}=\exp\left(-I\int_{t_0}^{t_0'}\mathrm{d}\mathscr{S}\right)\mathscr{D}x\qquad \mathscr{D}x=\prod_t\frac{1}{\sqrt{2\bar{m}_0}}\prod_i\frac{\mathrm{d}x'^i}{\sqrt{\mathrm{d}t}}$$

$$\begin{aligned}\hbar\,\mathrm{d}\mathscr{S}=&\frac{m_0}{2}\frac{(\mathrm{d}x^0)^2}{\mathrm{d}t}+\frac{m_0'}{4}\frac{(\mathrm{d}x^0)^3}{\mathrm{d}t}+\eta_0\frac{(\mathrm{d}x^0)^4}{\mathrm{d}t}+\varepsilon_0(\mathrm{d}x^0)^2+V\,\mathrm{d}t\\&-\sum_{k=1}^{N}\left[\frac{\alpha_k'\,\mathrm{d}x^0\,\mathrm{d}x^k}{\mathrm{d}t}+\frac{\alpha_k''(\mathrm{d}x^0)^2\,\mathrm{d}x^k}{2\,\mathrm{d}t}+\frac{\alpha_k'''(\mathrm{d}x^0)^3\,\mathrm{d}x^k}{6\,\mathrm{d}t}+\beta_k'x^k\,\mathrm{d}x^0\right.\\&\left.+(\partial_t\alpha_k+\gamma_k)\,\mathrm{d}x^k+\left(\partial_t\alpha_k'+\frac{\beta_k'+\gamma_k'}{2}\right)\mathrm{d}x^0\,\mathrm{d}x^k+(\partial_t\beta_k-\delta_k)x^k\,\mathrm{d}t\right]\\&+\sum_{k=1}^{N}\frac{(\mathrm{d}x^k)^2}{2\,\mathrm{d}t}+\tfrac{1}{2}\omega_k^2(x^k)^2\,\mathrm{d}t\end{aligned} \tag{7}$$

where $f'=\partial_0 f$ and

$$\eta_0=\frac{1}{12}\left(m_0''-\frac{\bar{m}_0'^2}{8\bar{m}_0}-\frac{1}{2}\sum_{k=1}^{N}\alpha_k''^2\right)\qquad\text{with}\qquad \bar{m}_0=m_0-\sum_{k=1}^{N}\alpha_k'^2=|m|$$

$$\varepsilon_0=\frac{\partial_t m_0}{4}-\frac{1}{2}\sum_{k=1}^{N}\beta_k''x^k.$$

As is apparent in this expression, coupling to the velocities results in the occurrence of new terms that vanish in the classical limit (when $\mathrm{d}x^i$ is of order $\mathrm{d}t$), but do contribute to the integral for a generic path (when $\mathrm{d}x^i$ is of order $\sqrt{\mathrm{d}t}$). These Itô terms (MacLaughlin and Schulman 1971), the coupling ones being proportional to α_k, β_k, γ_k, will be naturally dealt with in the following. Wick's rotation (5) will correspond to

$$t\to -\mathrm{i}t\qquad \beta_k\to\mathrm{i}\beta_k\qquad \gamma_k\to\mathrm{i}\gamma_k.$$

After integration on all variables $x^k(t)$, except for the initial and final time variables $x^k(t_0)$ and $x^k(t_0')$, the remaining reduced measure will provide the joint probability for the small system's variables at any time $x^0(t)$, and for the bath variables at initial and final times only x^k and x'^k:

$$\bar{\mathscr{P}}=\hat{\rho}(t_0',t_0)\bar{\mathscr{I}}$$

$$\bar{\mathscr{I}}=\int_{x^0(t),x^k(t_0')=x'^k,x^k(t_0)=x^k}\mathscr{I}=\exp\left(-I\int_{t_0}^{t_0'}\mathrm{d}\bar{\mathscr{S}}\right)\mathscr{D}x^0\prod_k\mathrm{d}x^k\,\mathrm{d}x'^k$$

$$\mathscr{D}x^0=\prod_t \frac{1}{\sqrt{2\bar{m}_0}}\frac{\mathrm{d}x'^0}{\sqrt{\mathrm{d}t}} \qquad \bar{m}_0=m_0-\sum_{k=1}^{N}\alpha_k'^2$$

$$I\int_{t_0}^{t_0'}\mathrm{d}\bar{\mathscr{S}}=\frac{1}{\hbar}\int_{t_0}^{t_0'}\left(\frac{\bar{m}_0}{2}\frac{(\mathrm{d}x^0)^2}{\mathrm{d}t}+\frac{\bar{m}_0'}{4}\frac{(\mathrm{d}x^0)^3}{\mathrm{d}t}+\bar{\eta}_0\frac{(\mathrm{d}x^0)^4}{\mathrm{d}t}+\frac{\dot{\bar{m}}_0}{4}(\mathrm{d}x^0)^2+\bar{V}\,\mathrm{d}t+\mathrm{d}q_0\right)$$
$$-\sum_{k=1}^{N}\ln G_k(x^k,x'^k,j_k) \qquad (8)$$

$$\bar{\eta}_0=\frac{1}{12}\left(\bar{m}_0''-\frac{\bar{m}_0'^2}{8\bar{m}_0}\right)$$

$$\bar{V}=V-\partial_t q_0-\tfrac{1}{2}\sum_{k=1}^{N}(\partial_t\alpha_k)^2 \qquad q_0'=-\sum_{k=1}^{N}\alpha_k'\partial_t\alpha_k$$

$$G_k(x,x',j)=\int_{x,x'}\exp\left[I\frac{1}{\hbar}\int_{t_0}^{t_0'}-\frac{1}{2}\left(\frac{(\mathrm{d}x-\mathrm{d}\alpha)^2}{\mathrm{d}t}+\omega_k^2x^2\,\mathrm{d}t\right)\right.$$
$$\left.+\left(\gamma+\tfrac{1}{2}(\mathrm{d}\beta+\mathrm{d}\gamma)\right)\mathrm{d}x+(\mathrm{d}\beta-\delta\,\mathrm{d}t)x\right]\mathscr{D}x$$

$$j=(\alpha,\beta,\gamma,\delta) \qquad j_k=(\alpha_k,\beta_k,\gamma_k,\delta_k)$$

where terms have been arranged together into total differentials:

$$\mathrm{d}\alpha=\alpha'\,\mathrm{d}x^0+\frac{\alpha''}{2}(\mathrm{d}x^0)^2+\partial_t\alpha\,\mathrm{d}t+\frac{\alpha'''}{6}(\mathrm{d}x^0)^3+\partial_t\alpha'\,\mathrm{d}x^0\,\mathrm{d}t$$

$$\mathrm{d}\beta=\beta'\,\mathrm{d}x^0+\frac{\beta''}{2}(\mathrm{d}x^0)^2+\partial_t\beta\,\mathrm{d}t$$

when developed up to contributing order.

Performing this integration and obtaining an explicit expression for the reduced measure will thus realise the second step of the program. This will provide the basis for a discussion of the nature of the process governing the small system, depending on the nature of the fluctuations in the initial and final states of the bath. At this point, this choice (of initial and final values of the bath variables) for boundary conditions is different from the usual one and might seem unsuited to (usual) physical situations. But, recalling the definition of the measure (4), this choice is dictated by the way the quantum state enters the path integral and thus completes the determination of its associated stochastic process. As will be discussed in the next section, the present derivation will nonetheless make contact with the standard time asymmetric treatment of the Langevin equation.

The solvability of the model, i.e. the possibility to realise explicitly the program's second step, results as usual (Ford *et al* 1965) from the choice of harmonic oscillators for representing the bath degrees of freedom. Here, the solvability takes the form of Gaussian integrals in the measure describing the bath degrees of freedom. It is well known that the result of this integration is given by the value of the Gaussian kernel at its extremum (with fixed boundary values: x at t_0 and x' at t_0'). As the Gaussian kernel has been obtained from the classical Lagrangian (where the x^0-dependent terms can be considered as source terms), the extremum identifies, for each oscillator x^k, with the classical solution running from x^k at t_0 to x'^k at t_0'. Thus, the stochastic treatment of the second step closely parallels the standard approach (Ford *et al* 1965).

However, because of the generic non-differentiability of the sample paths supporting the measure, care must be taken when determining the extremum from differential equations and, also, because of the stochastic nature of the source terms (depending on x^0), when computing the resulting contribution stochastic integrals must be used and the stochastic corrections to the classical Lagrangian (Itô terms) must not be ignored. Again, a safe procedure amounts to the following sequence: come back to the definition of the measure as a limit of its discretised version; then perform the integration; and only at the end take the continuous limit of the discretised result. With the convenient choice of the classical Lagrangian under its diagonal form with respect to the modes, one need only perform the integration on one mode, the final result being obtained by a straightforward product over all modes at the end. The Gaussian integral is performed in the appendix. Inserting for each mode k in (8), the corresponding expression for G_k provides the final analytic expression for the reduced measure $\bar{\mathscr{P}}$:

$$\begin{aligned}\bar{\mathscr{P}} &= \hat{\rho}(t_0', t_0)\exp\left(-\frac{1}{\hbar}[q_0(t_0') - q_0(t_0) + \mathscr{S}(x^0)]\right)\mathscr{D}x^0\\ &\quad\times\prod_{k=1}^{N}\exp\left(\frac{1}{\hbar}[\gamma_k'^0 x'^k - \gamma_k^0 x^k + Q_k(x^k - \alpha_k^0, x'^k - \alpha_k'^0, j_k)]\right)\mathrm{d}x^k\,\mathrm{d}x'^k\\ \mathscr{S}(x^0) &= I\int_{t_0}^{t_0'}\frac{\bar{m}_0}{2}\frac{(\mathrm{d}x^0)^2}{\mathrm{d}t} + \frac{\bar{m}_0'}{4}\frac{(\mathrm{d}x^0)^3}{\mathrm{d}t} + \bar{\eta}_0\frac{(\mathrm{d}x^0)^4}{\mathrm{d}t} + \frac{\dot{\bar{m}}_0}{4}(\mathrm{d}x^0)^2 + \bar{V}\,\mathrm{d}t\\ &\quad + S\int_{t_0}^{t_0'}\sum_{k=1}^{N}[\alpha_k\,\mathrm{d}j_k(t) - \tfrac{1}{2}\omega_k^2\alpha_k^2\,\mathrm{d}t] + \tfrac{1}{2}S\int_{t_0}^{t_0'}\int_{t_0}^{t_0'}\sum_{k=1}^{N}\sigma_k(t,s)\,\mathrm{d}j_k(t)\,\mathrm{d}j_k(s)\end{aligned}\tag{9}$$

where

$$\begin{aligned}Q_k(\chi, \chi', j_k) &= -\frac{\omega_k}{2}\left\{\frac{1+u_k^2}{1-u_k^2}[\chi'^2 + \chi^2] - 4\frac{u_k}{1-u_k^2}\chi'\chi\right\}\\ &\quad + \frac{2\omega_k}{1-u_k^2}[\chi'(r_k^- - u_k r_k^+) + \chi(r_k^+ - u_k r_k^-)]\end{aligned}$$

with

$$\begin{aligned}&\alpha_k(t_0) = \alpha_k^0 \qquad \alpha_k(t_0') = \alpha_k'^0 \qquad u_k = \mathrm{e}^{\omega_k(t_0 - t_0')}\\ &\mathrm{d}j_k(t) = \mathrm{d}(\gamma_k - \beta_k)(t) + (\omega_k^2\alpha_k + \delta_k)(t)\,\mathrm{d}t \qquad \gamma_k(t_0') = \gamma_k'^0 \qquad \gamma_k(t_0) = \gamma_k^0\\ &r_k^+ = S\int_{t_0}^{t_0'} -\frac{1}{2\omega_k}\mathrm{e}^{\omega_k(t_0 - s)}\,\mathrm{d}j_k(s) \qquad r_k^- = S\int_{t_0}^{t_0'} -\frac{1}{2\omega_k}\mathrm{e}^{\omega_k(s - t_0')}\,\mathrm{d}j_k(s)\\ &\sigma_k(t,s) = \frac{\mathrm{e}^{-\omega_k|t-s|}}{2\omega_k} - \frac{\mathrm{e}^{\omega_k(2t_0 - t - s)} + \mathrm{e}^{\omega_k(t+s-2t_0')} - u_k^2[\mathrm{e}^{\omega_k(t-s)} + \mathrm{e}^{\omega_k(s-t)}]}{2\omega_k(1-u_k^2)}.\end{aligned}\tag{10}$$

S stands for Stratonovich integrals; that is, by definition: for f and g two arbitrary diffusion variables

$$S\int_{t_0}^{t_0'} f\,\mathrm{d}g = \lim_{n\to\infty}\sum_{i=0}^{n-1} f\left(\frac{t_i + t_{i+1}}{2}\right)[g(t_{i+1}) - g(t_i)]$$

with the property that Stratonovich integrals preserve the differentiation chain rule: for any arbitrary diffusion variables f and g:

$$fg(t') - fg(t) = S\int_t^{t'} f\,\mathrm{d}g + S\int_t^{t'} g\,\mathrm{d}f.$$

Let us make here a few remarks. For definiteness, we have kept to Stratonovich integrals all through the previous derivation, but usual propagators are differentiable functions of time only, so that only one integral, namely $S\int_{t_0}^{t'_0}\alpha_k\,\mathrm{d}j_k(s)$, needs to be specified as Stratonovich. For all the other ones, any reference to Itô or Stratonovich can be ignored, as both prescriptions provide the same result in such cases. As usual in stochastic mechanics, one still needs to specify the quantum state (its transformed density matrix $\hat\rho(t'_0, t_0)$) in order to achieve the determination of the reduced measure and of the corresponding stochastic process for the small system x^0. But before discussing this question, we shall first derive some general consequences for the correlation functions, that will hold independently of the chosen quantum state.

2.3. *Correlation functions*

The generating functional for the correlation functions of the bath modes is straightforwardly obtained from the measure of the process (9), by substituting $\delta_k - \hbar h_k$ for δ_k. Hence, one easily deduces that the generating functional for the correlation functions of the bath modes, taken conditionally in the history of the small system and also in the initial and final values of the modes, takes the following expression:

$$\mathscr{G}(x^0, x^k, x'^k, h_k) = \exp\left(\sum_{k=1}^{N}\int_{t_0}^{t'_0} h_k(t)\tilde{x}^k(t)\,\mathrm{d}t + \frac{\hbar}{2}\int_{t_0}^{t'_0}\int_{t_0}^{t'_0} h_k(t)h_k(s)\sigma_k(t,s)\,\mathrm{d}t\,\mathrm{d}s\right)$$

where $\tilde{x}^k$ is the classical solution running from x^k at t_0 to x'^k at t'^0:

$$\tilde{x}^k(t) = S\int_{t_0}^{t'_0} G_k(t,s)\,\mathrm{d}j_k(s) + \alpha_k + \tilde{y}^k(t). \tag{11}$$

G_k is any propagator, and $\tilde{y}^k$ the corresponding free solution (see the appendix). One can use, for instance,

$$\tilde{x}^k(t) = -S\int_{t_0}^{t'_0}\sigma_k(t,s)\,\mathrm{d}j_k(s) + \alpha_k + \frac{\chi'^k - u_k\chi^k}{1-u_k^2}\,\mathrm{e}^{\omega_k(t-t'_0)} + \frac{\chi^k - u_k\chi'^k}{1-u_k^2}\,\mathrm{e}^{\omega_k(t_0-t)}$$

$$\chi^k = x^k - \alpha_k$$

but this is only a particular choice, as will be discussed in the next section ($\sigma_k(t,s)$ is the propagator which vanishes at t_0 and t'_0 (as previously defined in (10)); then the bath variables obey a Gaussian distribution (conditionally in $x^0(t)$ and x^k, x'^k), with one-point and two-point correlation functions characterised by

$$\langle x^k(t)\rangle_{x^0,x,x'} = \tilde{x}^k(t)$$

$$\langle x^k(t)x^{k'}(s)\rangle_{x^0,x,x'} = \tilde{x}^k(t)\tilde{x}^{k'}(s) + \hbar\delta_{kk'}\sigma_k(t,s).$$

As a result, the two-point correlation functions involving the bath modes will be determined by the correlation functions of the small system according to the following equation:

$$\langle x^k(t')x^i(t)\rangle = \int_{t_0}^{t'_0} G_k(t',s)[\partial_s\langle(\gamma_k-\beta_k)(s)x^i(t)\rangle + \langle(\omega_k^2\alpha_k+\delta_k)(s)x^i(t)\rangle]\,\mathrm{d}s$$
$$+\langle\alpha_k(t')x^i(t)\rangle + \langle\tilde{y}^k(t')x^i(t)\rangle + \hbar\delta_{ki}\sigma_k(t',t) \qquad (i=0,k'). \tag{12}$$

We shall restrict ourselves, in the following, to the case when the diffusion coefficients do not depend on the small system's variable:

$$\bar{m}'_0 = 0 \qquad m'_0 = \alpha''_k = 0.$$

Although such a restriction is not absolutely necessary, it has the advantage of leading to closed and simple expressions. In that case, and as a general consequence, the measure (7) of the diffusion process provides correlation functions which, besides (12), also satisfy

$$\begin{aligned}
&\partial_t \bar{m}_0 \partial_t \langle x^0(t) F(t')\rangle - \langle \bar{V}'(t) F(t')\rangle + \langle U_0(t) F(t')\rangle \\
&\qquad + \sum_{k=1}^{N} [\partial_{t''} \langle (\gamma_k' - \beta_k')(t) \chi^k(t'') F(t')\rangle|_{t''=t} - \langle (\delta_k' + \omega_k^2 \alpha_k') \chi^k(t) F(t')\rangle] \\
&\quad = -\hbar \delta(t-t') \langle F'(t)\rangle
\end{aligned}$$

$$U_0 = \sum_{k=1}^{N} \partial_t \alpha_k (\gamma_k' - \beta_k') - \alpha_k' \partial_t (\gamma_k - \beta_k) - \alpha_k' (\delta_k + \omega_k^2 \alpha_k) - \alpha_k \delta_k'$$

for any function F of x^0. For the sake of simplicity in the formula, F has been chosen to be function of x^0 taken at a single time t', but the same equations hold with F a general function of x^0 taken at arbitrary different times, provided a corresponding modification of the right-hand side. Then, inserting (12) one obtains that the correlations of the small system satisfy the following equations:

$$\begin{aligned}
&\partial_t \bar{m}_0 \partial_t \langle x^0(t) F(t')\rangle - \langle \bar{V}'(t) F(t')\rangle + \langle U_0(t) F(t')\rangle \\
&\qquad + \sum_{k=1}^{N} \int_{t_0}^{t_0'} \partial_t G_k(t,s) [\partial_s \langle (\gamma_k' - \beta_k')(t)(\gamma_k - \beta_k)(s) F(t')\rangle \\
&\qquad + \langle (\gamma_k' - \beta_k')(t)(\delta_k + \omega_k^2 \alpha_k)(s) F(t')\rangle] \\
&\qquad - G_k(t,s)[\partial_s \langle (\delta_k' + \omega_k^2 \alpha_k')(t)(\gamma_k - \beta_k)(s) F(t')\rangle \\
&\qquad + \langle (\delta_k' + \omega_k^2 \alpha_k')(t)(\delta_k + \omega_k^2 \alpha_k)(s) F(t')\rangle] \\
&\quad = \langle f(t) F(t')\rangle - \hbar \delta(t-t') \langle F'(t)\rangle
\end{aligned} \tag{13}$$

with

$$f(t) = \sum_{k=1}^{N} [\omega_k^2 \alpha_k' + \delta_k' + (\beta_k' - \gamma_k') \partial_t] \tilde{y}^k(t). \tag{14}$$

Let us remark that when choosing a stationary propagator ($G_k(t-s)$) these equations can be rewritten, after integration by parts, as

$$\begin{aligned}
&\partial_t \bar{m}_0 \partial_t \langle x^0(t) F(t')\rangle - \langle \bar{V}'(t) F(t')\rangle + \langle \bar{U}_0(t) F(t')\rangle - \langle \delta\mu(t) F(t')\rangle \\
&\qquad + \sum_{k=1}^{N} \langle (\gamma_k'^0 - \beta_k'^0)[(\gamma_k' - \beta_k')\partial_t - (\delta_k' + \omega_k^2 \alpha_k')](t) F(t')\rangle G_k(t - t_0') \\
&\qquad - \sum_{k=1}^{N} \langle (\gamma_k^0 - \beta_k^0)[(\gamma_k' - \beta_k')\partial_t - (\delta_k' + \omega_k^2 \alpha_k')](t) F(t')\rangle G_k(t - t_0) \\
&\quad = \langle f(t) F(t')\rangle - \hbar \delta(t-t') \langle F'(t)\rangle
\end{aligned}$$

with

$$\begin{aligned}
\mu(j) = +\frac{1}{2} \int_{t_0}^{t_0'} \int_{t_0}^{t_0'} \sum_{k=1}^{N} &G_k(t-s)[(\delta_k + \omega_k^2 \alpha_k)(t)(\delta_k + \omega_k^2 \alpha_k)(s) \\
&- \omega_k^2 (\gamma_k - \beta_k)(t)(\gamma_k - \beta_k)(s)] \\
&- \partial_t G_k(t-s)[(\gamma_k - \beta_k)(t)(\delta_k + \omega_k^2 \alpha_k)(s) \\
&- (\delta_k + \omega_k^2 \alpha_k)(t)(\gamma_k - \beta_k)(s)]\, \mathrm{d}t\, \mathrm{d}s
\end{aligned} \tag{15}$$

$$\delta\mu(t) = \frac{\delta\mu(j)}{\delta x^0(t)} \qquad \bar{U}_0 = U_0 + \sum_{k=1}^{N} (\beta_k' - \gamma_k')(\beta_k - \gamma_k).$$

The same integration by parts, performed directly on the measure $\mathscr{P}$, allows one to rewrite the effective action $\mathscr{S}(x^0)$ as the sum of boundary terms, depending only on the values at t_0' and t_0, a single integral, representing a local interaction, and a double integral, describing the self-interaction of the small system and being equal to $\mu(j)$.

Elimination of the bath degrees of freedom thus leads to an equation for the small system which has the familiar form of Langevin equations: a self-interaction term (μ), which *a priori* breaks time locality (memory effects), and a noise term (f), under the form of a correlation between the small system and a free field $\tilde{y}$. These equations being independent of the transformed density matrix entering the reduced measure (7), the particular quantum state will characterise the stochastic process and its correlation functions only through the boundary (initial or final) values that will be specified in order to complete the solution of the differential equations.

We shall now discuss the relation that exists between the correlation functions in the different representations. As recalled at the beginning of this section, the stochastic correlation functions identify with the time-ordered quantum correlation functions in a given state (after Wick's rotation (5)):

$$\langle\psi|T[\hat{x}^i(t')\hat{x}^j(t)]|\psi\rangle = \langle x^i(t')x^j(t)\rangle$$

with, in particular,

$$\langle\chi^k(t')\chi^{k'}(t)\rangle = \langle\tilde{\chi}^k(t')\tilde{\chi}^{k'}(t)\rangle + \hbar\delta_{kk'}\sigma_k(t', t) \qquad \text{where} \qquad \tilde{\chi}^k = \tilde{x}^k - \alpha_k.$$

Because of the differentiability of both the operators $\hat{\chi}^k(t)$ and the classical solutions $\tilde{\chi}^k(t)$, one also obtains

$$\langle\psi|T[\dot{\hat{\chi}}^k(t')\hat{\chi}^{k'}(t)]|\psi\rangle = \langle\dot{\tilde{\chi}}^k(t')\tilde{\chi}^{k'}(t)\rangle + \hbar\delta_{kk'}\partial_{t'}\sigma_k(t', t)$$
$$\langle\psi|T[\hat{\chi}^k(t')\dot{\hat{\chi}}^{k'}(t)]|\psi\rangle = \langle\tilde{\chi}^k(t')\dot{\tilde{\chi}}^{k'}(t)\rangle + \hbar\delta_{kk'}\partial_{t}\sigma_k(t', t)$$
$$\langle\psi|T[\dot{\hat{\chi}}^k(t')\dot{\hat{\chi}}^{k'}(t)]|\psi\rangle = \langle\dot{\tilde{\chi}}^k(t')\dot{\tilde{\chi}}^{k'}(t)\rangle + \hbar\delta_{kk'}[\partial_{t'}\partial_{t}\sigma_k(t', t) - \delta(t'-t)].$$

So, in particular, one will specify a given initial quantum state for the bath modes by the correlations of the operators $\hat{\chi}^k(t_0)$ $(=\hat{\chi}^k)$ and $\dot{\hat{\chi}}^k(t_0)$ $(=\dot{\hat{\chi}}^k)$ or, equivalently, by the correlations of the stochastic variables $\tilde{\chi}^k(t_0)$ $(=\chi^k)$ and $\dot{\tilde{\chi}}^k(t_0)$ $(=\dot{\chi}^k)$:

$$\begin{aligned}
&\langle\psi|\hat{\chi}^k\hat{\chi}^{k'}|\psi\rangle = \langle\tilde{\chi}^k\tilde{\chi}^{k'}\rangle \\
&\langle\psi|\dot{\hat{\chi}}^k\hat{\chi}^{k'}|\psi\rangle = \langle\dot{\tilde{\chi}}^k\tilde{\chi}^{k'}\rangle \\
&\langle\psi|\hat{\chi}^k\dot{\hat{\chi}}^{k'}|\psi\rangle = \langle\tilde{\chi}^k\dot{\tilde{\chi}}^{k'}\rangle + \hbar\delta_{kk'} \\
&\langle\psi|\dot{\hat{\chi}}^k\dot{\hat{\chi}}^{k'}|\psi\rangle = \langle\dot{\tilde{\chi}}^k\dot{\tilde{\chi}}^{k'}\rangle - \hbar\omega_k\frac{1+u_k^2}{1-u_k^2}\,\delta_{kk'}.
\end{aligned} \tag{16}$$

These properties will be used to make contact with the standard treatment of the quantum Langevin equation.

3. Noise and time locality

3.1. Noise associated with a bath state

The previous derivation of the reduced measure shows the same arbitrariness as that encountered in the standard elimination of the bath degrees of freedom, when solving the equations of motion. Indeed, the extremum of the Gaussian kernel (8) or, equivalently, the solution of the equations of motion (A4), is given by the general expression

$$\tilde{x}^k(t) = S\int_{t_0}^{t'_0} G_k(t, s)\, \mathrm{d}j_k(s) + \alpha_k + \tilde{y}^k(t) \tag{11}$$

where G_k is any propagator, that is, any function satisfying $(\partial_t^2 - \omega_k^2)G_k(t, s) = \delta(t - s)$, and $\tilde{y}^k$ is a free solution:

$$\tilde{y}^k(t) = a^k\, \mathrm{e}^{\omega_k(t-t'_0)} + a^{*k}\, \mathrm{e}^{\omega_k(t_0-t)}.$$

Of course, different choices for the propagator G_k will result in different free solutions $\tilde{y}^k$, so that (11) remains unchanged. For instance, the retarded propagator

$$G_k^r(t, s) = -\frac{1}{2\omega_k}\,\theta(t-s)[\mathrm{e}^{-\omega_k(t-s)} - \mathrm{e}^{\omega_k(t-s)}]$$

will be associated with the free solution as defined by

$$a^k = \frac{\chi'^k - u_k\chi^k + r^k}{1-u_k^2} \qquad a^{*k} = \frac{\chi^k - u_k\chi'^k + r^{*k}}{1-u_k^2}$$

with

$$r_k^* = -u_k r_k = -r_k^+ + u_k r_k^-$$

where $r_k^{+,-}$ are functions of x^0 defined as in (10). As a direct consequence, the correlation functions will also satisfy reduced equations of motion (13) which exhibit the same arbitrary choice as in (11). As usual, the free solution plays the role of an additive term in the equations of motion, resulting in the superposition of a noise component to the classical motion. Thus, the choice of a particular decomposition between propagator and free solution will result in a particular decomposition between self-interaction and noise. Such a choice will be dictated by the physical assumptions one will make on the fluctuations of the bath modes, as will be seen in the following. Let us also remark that the classical solution, here determined by its initial and final values, can also be specified in a completely equivalent way by its value and the value of its time derivative, both taken at initial (or final) time: indeed, $\tilde{\chi}^k$ is differentiable, with a time derivative equal to

$$\left.\frac{\mathrm{d}\tilde{\chi}^k}{\mathrm{d}t}\right|_{t_0} = -\omega_k \frac{(1+u_k^2)\chi^k - 2u_k\chi'^k - 2(r_k^+ - u_k r_k^-)}{1-u_k^2}$$

at the initial time, and:

$$\left.\frac{\mathrm{d}\tilde{\chi}^k}{\mathrm{d}t}\right|_{t'_0} = -\omega_k \frac{2u_k\chi^k - (1+u_k^2)\chi'^k + 2(r_k^- - u_k r_k^+)}{1-u_k^2}$$

at the final time. Thus, the reduced measure can be looked at in several ways: as a joint probability on the history of the small system and on either the values of the modes at initial and final times t_0 and t'_0, or the values of the modes and of their derivatives at initial time t_0, or else the values of the modes and of their derivatives at final time t'_0. The choice between these different ways to consider the reduced measure (which is arbitrary in principle) will also be dictated by the physical situation, and the corresponding assumptions on the fluctuations of the bath modes. In general, there will be two main physical situations. The first one, which is that of the usual quantum Langevin equation approach, corresponds to a small system which is coupled to a bath of definite statistical properties at initial time t_0. In this asymmetric situation one will want to relate the time evolution of the small system to the statistical distribution of the modes and of their derivatives at initial time t_0. Hence, the convenient decomposition will correspond to the retarded propagator, the related free field then being determined by the initial values of the modes and their derivatives at initial time t_0. The second situation is that of a stationary quantum state, where the small system is coupled to a bath with statistical properties remaining unchanged with time. That is, one will study the time evolution which can result for the small system when the modes at times t_0 and t'_0 are supposed to have the same definite statistical distribution. Then, the convenient choice will be a symmetric propagator. Such will be, in particular, the case of the vacuum, calling for the Feynman propagator and its related free field. Leaving the latter case to a forthcoming study, we shall deal here with the first situation only, and complete the study of the quantum Langevin equation in its stochastic representation, comparing it with its equivalent operatorial representation.

In order to obtain explicit and intrinsic expressions for the noise correlations, we shall make in the following the further assumption that the coupling is linear in the small system's variable:

$$\alpha_k = \alpha'_k x^0 \qquad \beta_k = \beta'_k x^0 \qquad \gamma_k = \gamma'_k x^0 \qquad \delta_k = \delta'_k x^0 \tag{17}$$

with α'_k, β'_k, γ'_k, δ'_k being constants, independent of x^0 and t. In that case, the noise, as defined in (14), will not depend on the small system's variable. Using the retarded propagator and its related free field $\tilde{y}^k_{\text{in}}$ defined by

$$a^k_{\text{in}} = \frac{1}{2u_k}\left(\chi^k + \frac{\dot{\chi}^k}{\omega_k}\right) \qquad a^{*k}_{\text{in}} = \frac{1}{2}\left(\chi^k - \frac{\dot{\chi}^k}{\omega_k}\right)$$

leads to the following input noise:

$$\begin{aligned} f_{\text{in}}(t) = \sum_{k=1}^{N} & [\omega_k^2\alpha'_k + \omega_k(\beta'_k - \gamma'_k) + \delta'_k]a^k_{\text{in}}\, \mathrm{e}^{\omega_k(t-t'_0)} \\ & + [\omega_k^2\alpha'_k - \omega_k(\beta'_k - \gamma'_k) + \delta'_k]a^{*k}_{\text{in}}\, \mathrm{e}^{\omega_k(t_0-t)}. \end{aligned}$$

One obtains the reduced measure (9) in the following form:

$$\bar{\mathscr{P}} = \hat{\rho}(t'_0, t_0)\exp\left(-\frac{1}{\hbar}\mathscr{S}^r(x^0)\right)\mathscr{D}x^0 \prod_{k=1}^{N} \exp\left(\frac{1}{\hbar}Q^{\text{in}}_k(\chi^k, \dot{\chi}^k, j_k)\right) \mathrm{d}x^k\, \mathrm{d}\dot{x}^k$$

$$\begin{aligned} Q^{\text{in}}_k(x, \dot{x}, j) = -\frac{\omega_k}{8u_k^2}\bigg[& (1-u_k^4)\left(x^2 + \frac{\dot{x}^2}{\omega_k^2}\right) + 2(1-u_k^2)^2\frac{\dot{x}}{\omega_k}x \\ & - 4\left((1+u_k^2)x + (1-u_k^2)\frac{\dot{x}}{\omega_k}\right)(r_k^+ + u_k r_k^-)\bigg] \\ & + \frac{\gamma_k'^0}{2u_k}\left((1+u_k^2)x + (1-u_k^2)\frac{\dot{x}}{\omega_k} - 2(r_k^+ - u_k r_k^-)\right) - \gamma_k^0 x \end{aligned}$$

$$\begin{aligned} \mathscr{S}^r(x^0) = I\int_{t_0}^{t'_0} \tfrac{1}{2}\bar{m}_0\frac{(\mathrm{d}x^0)^2}{\mathrm{d}t} + V\,\mathrm{d}t + S\int_{t_0}^{t'_0}\sum_{k=1}^{N}[\alpha'_k\,\mathrm{d}j_k(t) - \tfrac{1}{2}\omega_k^2\alpha_k'^2\,\mathrm{d}t] \\ - \tfrac{1}{2}S\int_{t_0}^{t'_0}\int_{t_0}^{t'_0}\sum_{k=1}^{N} G^r_k(t,s)\,\mathrm{d}j_k(t)\,\mathrm{d}j_k(s) \end{aligned}$$

with

$$\mathrm{d}j_k(t) = (\gamma'_k - \beta'_k)\,\mathrm{d}x^0 + (\delta'_k + \omega_k^2\alpha'_k)x^0\,\mathrm{d}t$$

leading to the following equations for the correlations of the small system:

$$\partial_t \bar{m}_0 \partial_t \langle x^0(t)F(t')\rangle - \langle V'(t)F(t')\rangle - \omega_0^2\langle x^0(t)F(t')\rangle - \int_{t_0}^{t'_0} \mu(t-s)\langle x^0(s)F(t')\rangle\,\mathrm{d}s$$
$$= \zeta(t-t_0)\langle x^0(t_0)F(t')\rangle + \langle f_{\mathrm{in}}(t)F(t')\rangle - \hbar\delta(t-t')\langle F'(t')\rangle \tag{18}$$

with

$$\omega_0^2 = \sum_{k=1}^{N} \frac{\bar{\mu}_k - \delta_k'^2}{\omega_k^2} \qquad \mu(\tau) = \sum_{k=1}^{N} \bar{\mu}_k G_k^r(\tau)$$
$$\bar{\mu}_k = (\delta'_k + \omega_k^2\alpha'_k)^2 - \omega_k^2(\beta'_k - \gamma'_k)^2 \tag{19}$$
$$\zeta(\tau) = \sum_{k=1}^{N} (\gamma'_k - \beta'_k)[(\gamma'_k - \beta'_k)\partial_\tau G_k^r(\tau) - (\delta'_k + \omega_k^2\alpha'_k)G_k^r(\tau)].$$

Let us remark that the original measure $\bar{\mathscr{P}}$ (9) is symmetric under time inversion, as can be seen by inspection of the self-interaction and noise contributions (only the symmetric part in front of $\mathrm{d}j(s)\,\mathrm{d}j(t)$ survives). Thus, the irreversible character shown by (18) and (19) is due to the particular choice of decomposition, retarded propagator and input noise, and will become effective when a particular measure is chosen for the input noise, corresponding to a particular state.

An identical parametrisation can also be used in the operator representation to solve the equivalent of equation (A4) for $\hat{x}^k(t)$:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \omega_k^2\right)(\hat{x}^k - \hat{\alpha}_k) - \hat{j}_k = 0 \qquad \text{with} \qquad \hat{j}_k = \dot{\hat{\gamma}}_k - \dot{\hat{\beta}}_k + \omega_k^2\hat{\alpha}_k + \hat{\delta}_k$$

under the form

$$\hat{x}^k = \hat{\alpha}_k + \int_{t_0}^{t'_0} G_k^r(t-s)\hat{j}^k(s)\,\mathrm{d}s + \hat{a}_{\mathrm{in}}^k\,\mathrm{e}^{\omega_k(t-t'_0)} + \hat{a}_{\mathrm{in}}^{+k}\,\mathrm{e}^{\omega_k(t_0-t)}$$

leading to the operatorial input noise (Nakazawa 1986)

$$\hat{f}_{\mathrm{in}}(t) = \sum_{k=1}^{N} [\omega_k^2\alpha'_k + \omega_k(\beta'_k - \gamma'_k) + \delta'_k]\hat{a}_{\mathrm{in}}^k\,\mathrm{e}^{\omega_k(t-t'_0)}$$
$$+ [\omega_k^2\alpha'_k - \omega_k(\beta'_k - \gamma'_k) + \delta'_k]\hat{a}_{\mathrm{in}}^{+k}\,\mathrm{e}^{\omega_k(t_0-t)}$$
$$\hat{a}_{\mathrm{in}}^k = \frac{1}{2u_k}\left(\hat{\chi}^k + \frac{\dot{\hat{\chi}}^k}{\omega_k}\right) \qquad \hat{a}_{\mathrm{in}}^{+k} = \frac{1}{2}\left(\hat{\chi}^k - \frac{\dot{\hat{\chi}}^k}{\omega_k}\right) \qquad \hat{\chi}^k = \hat{x}^k - \hat{\alpha}_k.$$

In any representation, the stochastic process and its correlation functions (obtained either from the reduced measure or from the operators) will only be determined when the quantum state is specified. The correlation functions being solutions of (18) will be determined when the correlations at the initial time are known, and in particular (for the bath modes) when the autocorrelations of the noise are given. Using (16), one easily obtains the following relation between the autocorrelations of the noise in the different representations:

$$\langle f_{\mathrm{in}}(t)f_{\mathrm{in}}(t')\rangle = \tfrac{1}{2}\langle\{\hat{f}_{\mathrm{in}}(t), \hat{f}_{\mathrm{in}}(t')\}\rangle - f_0(t-t') + \Sigma(t, t') \tag{20}$$

with

$$\Sigma(t, t') = \hbar \sum_{k=1}^{N} [\alpha'_k \partial^2_{t'} - (\beta'_k - \gamma'_k)\partial_{t'} + \delta'_k][\alpha'_k \partial^2_t + (\beta'_k - \gamma'_k)\partial_t + \delta'_k]$$
$$\times \left(\frac{1}{2\omega_k} e^{-\omega_k |t-t'|} - \sigma_k(t, t') \right)$$
$$f_0(t-t') = \hbar \sum_{k=1}^{N} \frac{\bar{\mu}_k}{4\omega_k} [e^{\omega_k(t-t')} + e^{\omega_k(t'-t)}].$$

One recovers for the noise autocorrelation function (up to a universal function $\Sigma(t, t') - f_0(t-t')$ which is independent of the bath state), the standard expression for the expectation value of the anticommutator of the noise (Ford and Kac 1986, Nakazawa 1986). Moreover, the universal function is the sum of a transient expression $\Sigma(t, t')$, which disappears when t_0 is sent to $-\infty$ and t'_0 to $+\infty$, and of an irreducible contribution which is precisely that of the vacuum, $f_0(t-t')$. So the quantum autocorrelation can be interpreted as the sum, up to a transient contribution, of the stochastic and the vacuum quantum autocorrelations. Let us recall that the expectation value of the commutator is also given by a universal function, independent of the bath state:

$$[\hat{f}_{\rm in}(t), \hat{f}_{\rm in}(t')] = \hbar \sum_{k=1}^{N} \frac{\bar{\mu}_k}{2\omega_k} [e^{\omega_k(t'-t)} - e^{\omega_k(t-t')}] \tag{21}$$

(in Euclidean space: $[\dot{x}^i, x^j] = -\hbar m^{ij}$ so that

$$\mu(t-t') = -\theta(t-t') \frac{1}{\hbar} [\hat{f}_{\rm in}(t), \hat{f}_{\rm in}(t')]$$

which is the usual fluctuation-dissipation relation).

To sum up, relations (20) and (21) can also be seen to restore for the noise (as defined in (14)), precisely in the limit of infinite t_0 and t'_0, the familiar relation between the stochastic correlations and their operatorial counterparts (time-ordered product of the quantum operators):

$$\langle \psi | T[\hat{f}_{\rm in}(t) \hat{f}_{\rm in}(t')] | \psi \rangle = \langle f_{\rm in}(t) f_{\rm in}(t') \rangle + G^f_0(t-t') - \Sigma(t, t') \tag{22}$$

where

$$G^f_0(\tau) = -\hbar \sum_{k=1}^{N} \bar{\mu}_k G^f_k(\tau)$$

is the Feynman propagator of the input noise in the vacuum state. Finally, Wick's rotation (5) must be performed on all previous expressions in order to recover the real quantum correlations.

3.2. Continuum limit and time locality

In the last step of the program, one wants to characterise the fluctuations of the process governing the time evolution of the small system in a rather simple way, i.e. one wants to exhibit an evolution which is local in time ('Markov property'). For that purpose, the limit of a bath consisting of a continuous infinity of modes will be taken. As is apparent in expressions like (15), which determines the correlation functions of the small system, the statistical properties in the limiting case will be determined by the mode dependence of the coupling, or more precisely by a function describing the spectral (frequency) dependence of the coupling constants. However, and quite generally, the effective self-interaction of the small system, as resulting from the coupling with the bath modes, will involve integrals that diverge and thus call for a renormalisation procedure. We shall follow the same procedure as in Ford *et al* (1985) and introduce a regularising high frequency cut-off (say Ω), in the form of a modified density for the bath modes, which will allow one to recover the correct coupling in its infinite limit ($\Omega \to \infty$). The renormalisation procedure, which will prescribe how to cope with the infinities appearing in the infinite cut-off limit and how to obtain finite physical expressions, will be defined later. Thus, the discrete summation over the bath modes will be replaced by the following continuous integral:

$$\sum_{k=1}^{N} \to \int_0^\infty \mathrm{d}\kappa\, \rho_\Omega(\kappa) \left[\frac{\mathrm{d}k}{\mathrm{d}\omega_k} = \rho_\Omega(\omega_k) = \left(\frac{\Omega^2}{\Omega^2 + \omega_k^2} \right)^n \rho(\omega_k) \right]$$

where, for convenience, the bath modes have been labelled by their frequency $\kappa = \omega_k$. One must also recall that all expressions will have to obey Wick's rotation (5), in order to identify with the quantum expressions. Then, defining quite generally for any arbitrary function g, a related retarded propagator by

$$G_g^{\mathrm{r}}(t) = \int_0^\infty \mathrm{d}\kappa\, g(\kappa) \frac{1}{2\kappa} \theta(t) [\mathrm{e}^{-\mathrm{i}\kappa t} - \mathrm{e}^{\mathrm{i}\kappa t}]$$

so that

$$\sum_{k=1}^{N} g(\omega_k) G_k^{\mathrm{r}} \to G_{\rho_\Omega g}^{\mathrm{r}}$$

the self-interaction μ in the time evolution for the small system (15) will be determined by the retarded propagator $\mu = G_{\bar{\mu}}^{\mathrm{r}}$ with

$$\bar{\mu}(\omega_k) = \rho(\omega_k) \bar{\mu}_k \qquad \bar{\delta}^2(\omega_k) = \rho(\omega_k) \delta_k'^2.$$

In order to compute this propagator explicitly, let us introduce its Fourier transform, as defined by

$$\tilde{G}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} G(t)$$

so that

$$\tilde{G}_k^{\mathrm{r}}(\omega) = \lim_{\varepsilon \to 0} \frac{\mathrm{i}}{(\omega + \mathrm{i}\varepsilon)^2 - \omega_k^2}$$

and

$$\tilde{G}_g^{\mathrm{r}}(\omega) = \lim_{\varepsilon \to 0} \int_0^\infty \mathrm{d}\kappa\, g(\kappa) \frac{\mathrm{i}}{(\omega + \mathrm{i}\varepsilon)^2 - \kappa^2}$$

are analytic functions in the complex upper half plane. Locality in time will now be shown to result from the following behaviour of the coupling constants (μ_{d} and μ_{r} being arbitrary parameters):

$$\bar{\mu}(\kappa) = \frac{2}{\pi} [\mu_{\mathrm{d}} \kappa^2 + \mu_{\mathrm{r}} \kappa^4]$$

or

$$\rho(\omega_k)[(\delta'_k+\omega_k^2\alpha'_k)^2+\omega_k^2(\gamma'_k-\beta'_k)^2]=\frac{2}{\pi}[\mu_d\omega_k^2+\mu_r\omega_k^4]. \tag{23}$$

Taking n equal to 2 in $\tilde{\mu}_\Omega(\kappa)=[\Omega^2/(\Omega^2+\kappa^2)]^n\tilde{\mu}(\kappa)$, one obtains

$$\begin{aligned}\tilde{\mu}(\omega)\to\tilde{G}^{\mathrm{r}}_{\tilde{\mu}_\Omega}(\omega)&=-\frac{\mathrm{i}}{2}\left(\mu_r\Omega^3+(\mu_d+\mu_r\omega^2)\Omega+\mathrm{i}\frac{\Omega(2\Omega-\mathrm{i}\omega)}{(\Omega-\mathrm{i}\omega)^2}(\mu_d\omega^2+\mu_r\omega^4)\right)\\&=-\frac{\mathrm{i}}{2}\left(\mu_d\Omega+\mu_r\Omega^3+2\mathrm{i}\mu_d\omega+\mu_r\Omega\omega^2+2\mathrm{i}\mu_r\omega^3\right)+\mathrm{o}\left(\frac{1}{\Omega}\right)\end{aligned}$$

or else

$$\mu_\Omega(t)=-\frac{\mathrm{i}}{2}(\mu_d\Omega+\mu_r\Omega^3)\delta(t)-\mathrm{i}\mu_d\dot{\delta}(t)+\frac{\mathrm{i}}{2}\mu_r\Omega\ddot{\delta}(t)+\mathrm{i}\mu_r\dddot{\delta}(t)+\mathrm{o}\left(\frac{1}{\Omega}\right).$$

Noting that ω_0^2 becomes

$$\omega_0^2=\tfrac{1}{2}(\mu_d\Omega+\mu_r\Omega^3)-K_0\qquad\text{with}\qquad K_0=\int_0^\infty\left(\frac{\Omega^2}{\Omega^2+\kappa^2}\right)^2\frac{\bar{\delta}^2(\kappa)}{\kappa^2}\,\mathrm{d}\kappa$$

the self-interaction reduces to

$$\mu_\Omega(t)+\mathrm{i}\omega_0^2\delta(t)=-\mathrm{i}K_0\delta(t)-\mathrm{i}\mu_d\dot{\delta}(t)+\frac{\mathrm{i}}{2}\mu_r\Omega\ddot{\delta}(t)+\mathrm{i}\mu_r\dddot{\delta}(t)+\mathrm{o}\left(\frac{1}{\Omega}\right)$$

and the time evolution of the correlation functions of the small system becomes

$$\begin{aligned}&[\partial_t(\bar{m}_0+\tfrac{1}{2}\mu_r\Omega)\partial_t+\mu_d\partial_t-\mu_r\partial_t^3]\langle x^0(t)F(t')\rangle+\langle[V'(t)-K_0x^0(t)]F(t')\rangle\\&\quad=\zeta(t-t_0)\langle x^0(t_0)F(t')\rangle-\langle f_{\mathrm{in}}(t)F(t')\rangle+\mathrm{i}\hbar\delta(t-t')\langle F'(t')\rangle.\end{aligned} \tag{24}$$

These equations show the desired property: the time evolution has become local in the continuum limit; the previous integral involving a summation over the past has disappeared, leaving a pure differential equation, relating quantities taken all at the same time. This is sometimes referred to in the literature as a 'Markov' property, because of the memory loss in the equations of motion. However, it must be emphasised that the associated stochastic process (the reduced process), in general, is not Markovian (as can be easily seen, for instance, in the case when the small system is simply a harmonic oscillator).

These equations also show that, in order to keep a physical meaning to their solution, one is led to assume that the diffusion constant $\bar{m}_0$, and the potential V, will also contain a cut-off dependence, so that the coefficients in the equations of motion (24) have a finite limit when the cut-off is sent to infinity. To complete the determination of these equations, one will have to prescribe the physical values of two corresponding constants, like for instance the mass of the particle and the quadratic part of the potential. Hence, according to the type of coupling, two renormalisations might be necessary: one for the potential, but also one modifying the diffusion coefficient (wavefunction renormalisation). The case $\mu_r=0$ provides the usual treatment of dissipation alone, by coupling to a bath of harmonic oscillators (Ford and Kac 1986, Nakazawa 1986, Caldeira and Leggett 1981), and corresponds to

$$\alpha'_k=\beta'_k=\gamma'_k=0\qquad\rho(\omega_k)\delta'^2_k\sim\Gamma\omega_k^2\qquad(\mu_d\sim\Gamma\text{ and }\mu_r=0,\ K_0\sim\Omega).$$

In those cases, radiation damping is not present and renormalisation only affects the potential. The absence of wavefunction renormalisation implies that the canonical commutation rules are preserved, although the system has become dissipative (Dekker

1986). Indeed, the commutation relation between position and velocity (or momentum) is determined by the diffusion coefficient (Davidson 1979)

$$[\hat{x}^0, \dot{\hat{x}}^0] = \frac{2}{\nu_0} = \frac{\hbar}{\bar{m}_0}$$

so that, in that case, the free system and the interacting system do have the same canonical commutation rules. However, it must be emphasised that this property fails as soon as the coupling to the bath modes involves a frequency dependence of a higher degree as, for instance, by coupling through the velocity or to a higher spectral density of modes, only because renormalisation makes $\bar{m}_0$ go to infinity. Such will be the case for a particle which is coupled to the electromagnetic field, in the dipole approximation (Ford *et al* 1985):

$$\alpha'_k = 0, \beta'^2_k \sim e^2, \gamma'_k = \delta'_k = 0 \qquad \text{and} \qquad \rho(\omega_k) \sim \rho_0 \omega_k^2$$

so that $\mu_d = 0$, $\mu_r \sim \rho_0 e^2$ and $K_0 = 0$.

Many other types of coupling can be envisaged, that will combine couplings involving position and velocity of the small system to positions and velocities of the modes, and which will preserve the locality of the equations of motion in the continuum limit, as soon as relations (17) and (23) are satisfied. Coupling to the momentum can also be dealt with in the same completely stochastic way (translating to the Hamiltonian approach the Lagrangian treatment (Gardiner and Collett 1985)), as long as one assumes a quadratic coupling of the small system to a continuum of harmonic oscillators.

4. Conclusion

The purely stochastic approach, by using only measures and commuting variables, already presents several advantages for the quantum Langevin equation itself. The first one lies in the path integral representation, which provides the common formalism required for treating purely quantum effects, such as quantum tunnelling, mixed with statistical effects, such as dissipation. By means of diffusion processes, and of the associated stochastic calculus, this representation can be made sufficiently rigorous and general to cover the treatment of a wide variety of couplings, involving the velocities of the small system and the bath modes, and arbitrary dependences on the small system's position. Such freedom should be helpful when discussing in a realistic way the effects of environment on quantum tunnelling occurring, in particular, in macroscopic devices (Caldeira and Leggett 1983, Ford *et al* 1988). By escaping the problem of operator ordering and the corresponding ambiguities in the choice of the Hamiltonian, the stochastic approach provides a way to go beyond usual approximations (like the dipole one (Ford *et al* 1985)). Quite generally, commutativity characterises a representation of the quantum Langevin equation which is very similar to its classical counterpart. This opens an alternative way for, in analogy with the classical situation, relating this equation to a quantum extension of stochastic calculus. Ordinary commuting variables can still be used, instead of operatorial extensions, the difference with the classical case lying in the noise correlation functions, which depend on the quantum state.

The stochastic representation also provides a means for re-analysing old questions about quantum mechanics from a new point of view. The diffusion processes used in this representation as a general frame preserve time reversibility, as required by quantum mechanics (Nelson 1966). Time irreversibility, exhibited by a dissipative evolution of the small system for instance, has its origin in a particular choice of boundary conditions, like retarded propagator and input noise. This situation is identical to the familiar one which prevails between reversible classical mechanics and irreversible thermodynamics (Chandrasekhar 1943). That reversible elementary interactions might result in some cases in causal, and thus irreversible, macroscopic ones, such as the interactions with the bath, is not contradictory and can be understood in the same way as Feynman and Wheeler (1949) have shown for the electromagnetic field with the absorber theory. By referring explicitly to a bath with a large (infinite) number of degrees of freedom, one can raise the apparent incompatibility between diffusion processes and quantum mechanics, which seems to occur when considering the small system alone. This formalism provides an alternative means of discussing the respective characteristics of quantum and dissipative processes based on processes of the same nature (with different boundary conditions) and thus appears well suited for analysing the heuristic foundations of stochastic mechanics, such as the background field hypothesis (Nelson 1985). The limit of a continuum of modes, with a given spectral density, leads in both classical and quantum situations to the disappearance of memory effects, that is to time locality. However, the statistical independence which is characteristic of Markov processes is only obtained for the noise associated with particular states, such as in the high-temperature limit. This is another hint that the difference between quantum and classical stochastic processes lies in the choice of the statistical properties of the noise induced by the bath, rather than in the nature of the time evolution.

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Appendix

The Gaussian integral given in (8) is first discretised as

$$\Delta t_i = t_{i+1} - t_i = \frac{t_0' - t_0}{n} \qquad \Delta x_i = x_{i+1} - x_i \qquad x_0 = x,\ x_n = x'$$

$$G(x, x', j) = \lim_{n\to\infty} \int \exp\left(-\frac{1}{\hbar} \sum_{i=0}^{n-1} \Delta \mathscr{S}_i \right) \prod_{i=0}^{n-2} \frac{\mathrm{d}x_{i+1}}{\sqrt{\Delta t_i}} \tag{A1}$$

$$\Delta \mathscr{S}_i = \frac{1}{2}\left[\frac{(\Delta x_i - \Delta\alpha_i)^2}{\Delta t_i} + \omega^2(x_i)^2 \Delta t_i \right] - \tfrac{1}{2}(\gamma_i + \gamma_{i+1})\Delta x_i - \tfrac{1}{2}(x_i + x_{i+1})[\Delta\beta_i - \delta_i \Delta t_i] + \mathrm{o}(\Delta t_i).$$

Taking the value of the latter at its extremum $\tilde{x}_i$ one obtains

$$\begin{aligned}
\sum_{i=0}^{n-1} \Delta \mathscr{S}_i(\tilde{x}) = {} & \tfrac{1}{2}(\tilde{x}_n - \alpha_n) \frac{\tilde{x}_n - \alpha_n - \tilde{x}_{n-1} + \alpha_{n-1}}{\Delta t_{n-1}} - \tfrac{1}{2}(\tilde{x}_0 - \alpha_0) \frac{\tilde{x}_1 - \alpha_1 - \tilde{x}_0 + \alpha_0}{\Delta t_0} \\
& - \tfrac{1}{2}(\gamma_n + \gamma_{n-1} + \beta_n - \beta_{n-1})\tilde{x}_n + \tfrac{1}{2}(\gamma_1 + \gamma_0 + \beta_1 - \beta_0)\tilde{x}_0 \\
& + \tfrac{1}{4} \sum_{i=0}^{n-1} (\tilde{x}_i + \alpha_i)[\gamma_{i+1} - \gamma_{i-1} - \beta_{i+1} + \beta_{i-1} + 2(\delta_i + \omega^2\alpha_i)\Delta t_i] \\
& - 2\omega^2 \alpha_i^2 \Delta t_i + \mathrm{o}(1).
\end{aligned} \tag{A2}$$

The continuum limit then provides an explicit form for G in the following expression:

$$G(x, x', j) = \exp \frac{1}{\hbar} \left(T(t_0') - T(t_0) - \tfrac{1}{2} S \int_{t_0}^{t_0'} (\tilde{x} + \alpha)[\mathrm{d}(\gamma - \beta) + (\delta + \omega^2 \alpha)\, \mathrm{d}t] + \omega^2 \alpha^2\, \mathrm{d}t \right) \tag{A3}$$

with

$$T(t) = -\tfrac{1}{2}(\tilde{x} - \alpha) \frac{\Delta(\tilde{x} - \alpha)}{\Delta t} + \gamma \tilde{x}$$

where S stands for Stratonovich integrals and $\tilde{x}$ for the continuum limit of the extremum, that is the classical solution: indeed, one can easily see that $\tilde{x} - \alpha$ is differentiable (hence T has a limit), and so is $(\mathrm{d}/\mathrm{d}t)(\tilde{x} - \alpha) - \gamma + \beta$, so that $\tilde{x}$ is solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathrm{d}}{\mathrm{d}t} (\tilde{x} - \alpha) - \gamma + \beta \right) - \omega^2 \tilde{x} - \delta = 0 \tag{A4}$$

with $\tilde{x}(t_0') = x'$; $\tilde{x}(t_0) = x$. The latter can easily be obtained using the fact that Stratonovich integrals preserve the differentiation chain rule. Then, the general solution of (A4) is given by

$$\tilde{x}(t) = \alpha(t) + \tilde{y}(t) + S \int_{t_0}^{t_0'} G(t, s)\, \mathrm{d}j(s) \tag{A5}$$

where G is a propagator, that is a solution of

$$[\partial_t^2 - \omega^2] G(t, s) = \delta(t - s) \qquad \text{and} \qquad \mathrm{d}j(s) = \mathrm{d}(\gamma - \beta) + (\omega^2 \alpha + \delta)\, \mathrm{d}s$$

and where $\tilde{y}$ is an arbitrary free solution:

$$\tilde{y} = a\, \mathrm{e}^{\omega(t - t_0')} + a^* \, \mathrm{e}^{\omega(t_0 - t)}.$$

Fixing the boundary values

$$x(t_0) = x \qquad (\alpha(t_0) = \alpha^0;\ \gamma(t_0) = \gamma^0)$$

$$x(t_0') = x' \qquad (\alpha(t_0') = \alpha'^0;\ \gamma(t_0') = \gamma'^0)$$

determines $\tilde{y}$ and thus completes the determination of the solution $\tilde{x}$. Letting: $u = \mathrm{e}^{\omega(t_0 - t_0')}$ and

$$r^+ = S \int_{t_0}^{t_0'} -\frac{1}{2\omega} \mathrm{e}^{\omega(t_0 - s)}\, \mathrm{d}j(s) \qquad r^- = S \int_{t_0}^{t_0'} -\frac{1}{2\omega} \mathrm{e}^{\omega(s - t_0')}\, \mathrm{d}j(s)$$

and replacing $\tilde{x}$ by its explicit expression (A5) in (A3), provides the final result for the elementary Gaussian integral:

$$G(x, x', j) = \exp \frac{1}{\hbar} [\gamma'^0 x' - \gamma^0 x + Q(x - \alpha^0, x' - \alpha'^0, j) + R(j)]$$

with

$$Q(x, x', j) = -\frac{\omega}{2} \left\{ \frac{1 + u^2}{1 - u^2} [x'^2 + x^2] - 4 \frac{u}{1 - u^2} x' x \right\} + \frac{2\omega}{1 - u^2} [x'(r^- - u r^+) + x(r^+ - u r^-)]$$

$$R(j) = -S \int_{t_0}^{t_0'} \alpha\, \mathrm{d}j(s) - \tfrac{1}{2}\omega^2 \alpha\, \mathrm{d}s - \tfrac{1}{2} S \int_{t_0}^{t_0'} \int_{t_0}^{t_0'} \sigma(t, s)\, \mathrm{d}j(t)\, \mathrm{d}j(s)$$

where

$$\sigma(t,s)=\frac{e^{-\omega|t-s|}}{2\omega}-\frac{e^{\omega(2t_0-t-s)}+e^{\omega(t+s-2t_0')}-u^2[e^{\omega(t-s)}+e^{\omega(s-t)}]}{2\omega(1-u^2)}.$$

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